COUNTING FUNCTION OF CHARACTERISTIC VALUES AND MAGNETIC RESONANCES

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ABSTRACT. We consider the meromorphic operator-valued function $I - K(z) = I - A(z)/z$ where A is holomorphic on the domain $\mathcal{D} \subset \mathbb{C}$, and has values in the class of compact operators acting in a given Hilbert space. Under the assumption that $A(0)$ is a selfadjoint operator which can be of infinite rank, we study the distribution near the origin of the characteristic values of $I - K(z)$, i.e. the complex numbers $w \neq 0$ for which the operator $I - K(w)$ is not invertible, and we show that generically the characteristic values of $I - K$ converge to 0 with the same rate as the eigenvalues of $A(0)$.

We apply our abstract results to the investigation of the resonances of the operator $H = H_0 + V$ where H_0 is the shifted 3D Schrödinger operator with constant magnetic field of scalar intensity $b > 0$, and $V : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is the electric potential which admits a suitable decay at infinity. It is well known that the spectrum $\sigma(H_0)$ is purely absolutely continuous, coincides with $[0, +\infty[$, and the so-called Landau levels $2bq$ with integer $q \geq 0$, play the role of thresholds in $\sigma(H_0)$. We study the asymptotic distribution of the resonances near any given Landau level, and under generic assumptions obtain the main asymptotic term of the corresponding resonance counting function, written explicitly in the terms of appropriate Toeplitz operators.

1. INTRODUCTION

It is well known that several spectral problems for unbounded operators can be reduced to the study of a compact-operator-valued function $K(z)$. Generally, a complex number z in a domain $\mathcal{D}$ is an eigenvalue (or a resonance) of an operator H if and only if $I - K(z)$ is not invertible where $z \longmapsto K(z)$ is holomorphic on $\mathcal{D}$ with value in $\mathcal{S}_\infty$, the space of compact operators. For example, under suitable assumptions, according to the Birman–Schwinger principle, the study of the eigenvalues of $H = H_0 + M^*M$ can be related to the compact operator $K(z) = -M(H_0 - z)^{-1}M^*$ (see [5], [6], [15], [21], [22], [26], [34], [36], [37], [38]). For a more general Birman–Schwinger principle for non-selfajoint operators we refer to [16]. Similarly, the resonances for $H = H_0 + V$, a perturbation of a free Hamiltonian H_0, can be analyzed by studying the invertibility of $I - K(z)$ with $K(z)$ a compact operator (see [9], [12], [13], [19], [28], [35], [39], [40], [41], [45]). For example, for perturbations of Schrödinger operators H_0 by exponentially decreasing potentials V, thanks to a resolvent equation like (6.1), we can choose $K(z) = -\operatorname{sign}(V)|V|^{\frac{1}{2}}(H_0 - z)^{-1}|V|^{\frac{1}{2}}$. In other situations, $K(z)$ is constructed by more sophisticated methods, like Grushin problems or by a representation formula of the scattering matrix.

In what follows, as in [18], for $z \longmapsto K(z)$ holomorphic on $\mathcal{D}$ with values in $\mathcal{S}_\infty$, we will say that a complex number w is a *characteristic value* of $I - K(\cdot)$ if $I - K(w)$ is not invertible.

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According to the analytic Fredholm theorem, if for some $z_0 \in \mathcal{D}$ the operator $I - K(z_0)$ is invertible, then $I - K(\cdot)$ has a discrete set of characteristic values in $\mathcal{D}$. However, these characteristic values could accumulate at some point of the boundary $\partial\mathcal{D}$. For example, if A_0 is a selfadjoint compact operator of infinite rank, then the characteristic values of $I - A_0/z$ in $\mathbb{C} \setminus \{0\}$ are the eigenvalues of A_0 which accumulate at 0. If $z \longmapsto K(z)$ is holomorphic on a domain $\mathcal{D}$, then the number of characteristic values of $I - K$ in each compact subset of $\mathcal{D}$ is finite. This property still holds true if $z \longmapsto K(z)$ is finite meromorphic on $\mathcal{D}$ (see Section 2, Proposition 2.3 or [17, Proposition 4.1.4]). For example, we meet this case within the context of the investigation of the resonances for the 1D Schrödinger operator (see [12]).

In this paper, we consider the case where

$$I - K(z) = I - \frac{A(z)}{z}, \tag{1.1}$$

with $z \longmapsto A(z) : \mathcal{D} \longrightarrow \mathcal{S}_\infty$ holomorphic on a domain $\mathcal{D} \subset \mathbb{C}$ containing 0, and $A(0)$ selfadjoint. As mentioned above, $K(z)$ has typically the structure of a sandwiched resolvent of the unperturbed operator which explains its form defined in (1.1) where the factor $1/z$ models, after an appropriate change of the variables, the threshold singularity of $(H_0 - z)^{-1}$. An important specific feature of the operators we consider, is the fact that $A(0)$ can be of infinite rank. Many new phenomena described in the present article are due to this property. One encounters a similar situation when one studies the scattering poles on asymptotically hyperbolic manifolds (see [19]), or the resonances of the magnetic Schrödinger operator in $\mathbb{R}^3$ (see [7]). This type of problem could also arise for the investigation of resonances near thresholds for other magnetic Hamiltonians like those of [2], [20], [32], [43].

First, we consider the asymptotic distribution near the origin of the characteristic values of $I - A(z)/z$. The natural intuition is that these characteristic values accumulate at 0 with the same rate as the spectrum of $A(0)$, but since only $A(0)$ is assumed to be selfadjoint, some pseudospectral phenomena could perturb this conjecture. We describe situations where this intuition is really valid (see Section 3). Then we apply our abstract results to the study of the distribution of resonances near the spectral thresholds for the shifted 3D Schrödinger operators with constant magnetic field of strength $b > 0$, pointing at the x_3-direction:

$$H(b,V) := \Big(D_1 + \frac{b}{2}x_2\Big)^2 + \Big(D_2 - \frac{b}{2}x_1\Big)^2 - b + D_3^2 + V, \qquad D_j := -i\frac{\partial}{\partial x_j}. \tag{1.2}$$

We regard this operator as one of the main sources of motivation for the article, and hence we would like to discuss it in more detail. Set $X_\perp = (x_1, x_2) \in \mathbb{R}^2$. Using the representation $L^2(\mathbb{R}^3) = L^2(\mathbb{R}^2_{X_\perp}) \otimes L^2(\mathbb{R}_{x_3})$, we find that

$$H_0 := H(b,0) = H_{\text{Landau}} \otimes I_3 + I_\perp \otimes \Big(-\frac{\partial^2}{\partial x_3^2}\Big) \tag{1.3}$$

where

$$H_{\text{Landau}} := \Big(D_1 + \frac{b}{2}x_2\Big)^2 + \Big(D_2 - \frac{b}{2}x_1\Big)^2 - b, \tag{1.4}$$

is the shifted Landau Hamiltonian, selfadjoint in $L^2(\mathbb{R}^2)$, and I_3 and $I_\perp$ are the identity operators in $L^2(\mathbb{R}_{x_3})$ and $L^2(\mathbb{R}^2_{X_\perp})$ respectively. It is well known that the spectrum of H_{Landau} consists of the so-called Landau levels $2bq$, $q \in \mathbb{N} := \{0, 1, 2, \dots\}$, and $\dim \operatorname{Ker}(H_{\text{Landau}} - 2bq) = \infty$. Consequently,

$$\sigma(H_0) = \sigma_{\text{ac}}(H_0) = [0, +\infty[,$$

and we conclude that the Landau levels play the role of thresholds in the spectrum of H_0. Since the "transversal" operator $H_{\rm Landau}$ in (1.3) has a purely point spectrum, and its eigenvalues form a discrete subset of $\mathbb{R}$ while the spectrum of the "longitudinal" operator $-\frac{\partial^2}{\partial x_3^2}$ is purely absolutely continuous, the structure of H_0 is quite close to the one of the (unperturbed) quantum waveguide Hamiltonians. The study of the resonances for perturbations of such quantum waveguides and their generalizations has a rich history (see e.g. [1], [8], [10], [44]). The novelty of the results obtained in the present article as well as in its predecessor [7] is related to the fact that in the case of the operator $H(b,0)$ the spectral thresholds (i.e. the eigenvalues of the transversal operator $H_{\rm Landau}$ in (1.3)) are of infinite multiplicity; this corresponds to the fact that rank $A(0) = \infty$ in the case of the operator in (1.1) (see below (6.2) for the explicit expression of the operator A arising in the study of the resonances accumulating at the qth Landau level).

Assume now that the multiplier by the electric potential $V : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is relatively compact with respect to H_0. It is known that if V satisfies the estimate

$$V(\mathbf{x}) \leq -C\mathbb{1}_U(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^3, \tag{1.5}$$

where $C > 0$ and $U \subset \mathbb{R}^3$ is an open non empty set, the operator $H(b,V)$ has an infinite negative discrete spectrum (see e.g. [3, Theorem 1.5]). Next, if V is axisymmetric, i.e. depends only on $|X_\perp|$ and x_3, and satisfies (1.5), then below each Landau level $2bq$, $q \in \mathbb{N}$, the operator $H(b,V)$ has at least one eigenvalue which for all sufficiently large q is embedded in the essential spectrum (see [3, Theorem 1.5]). Finally, if V is axisymmetric and satisfies

$$V(\mathbf{x}) \leq -C\mathbb{1}_W(X_\perp)(1+|x_3|)^{-m_3}, \qquad \mathbf{x} = (X_\perp, x_3) \in \mathbb{R}^3, \tag{1.6}$$

where $C > 0$, $m_3 \in (0,2)$ and $W \subset \mathbb{R}^2$ is an open non empty set, then there exists an infinite series of eigenvalues of $H(b,V)$ below each Landau levels $2bq$ (see [30], [31]). Further, in [11] it was supposed that V is continuous, has a definite sign, does not vanish identically, and satisfies

$$V(\mathbf{x}) = \mathcal{O}\big((1+|X_\perp|)^{-m_\perp}(1+|x_3|)^{-m_3}\big), \qquad \mathbf{x} = (X_\perp, x_3) \in \mathbb{R}^3, \tag{1.7}$$

and it was shown that the Krein spectral shift function associated with the operator pair $(H(b,V), H(b,0))$ has singularities at the Landau levels. All these properties suggest that for generic V there could be an accumulation of resonances of $H(b,V)$ at the Landau levels.

In the present article we assume that V is Lebesgue measurable, and satisfies

$$V(\mathbf{x}) = \mathcal{O}\big((1+|X_\perp|)^{-m_\perp}\exp(-N|x_3|)\big), \qquad \mathbf{x} = (X_\perp, x_3) \in \mathbb{R}^3, \tag{1.8}$$

with $m_\perp > 0$, $N > 0$. In [7], we defined the resonances of $H(b,V)$ under this assumption, and gave an upper bound on their number at a distance $r \searrow 0$ of the Landau levels. Moreover, for $H(b,eV)$ with e sufficiently small and V of definite sign, compactly supported (or decreasing like a Gaussian function), we stated a lower bound. Here, in Section 6, we obtain the asymptotic behavior of the counting function of the magnetic resonances near the Landau levels for V of definite sign satisfying the estimate (1.8) and for every $e \in \mathbb{R} \setminus \mathcal{E}$ where $\mathcal{E}$ is a discrete set of $\mathbb{R}$. To our best knowledge, it is the first result giving such asymptotic behaviour for counting functions of magnetic resonances (and maybe, more generally for resonances which accumulate at a spectral threshold).

Apart from the applications in the mathematical theory of resonances for quantum Hamiltonians, we hope that our research could turn out to be useful for the better understanding of the so called magnetic Feshbach resonances which play an important role in the modern

theoretical physics, in particular the theory of Bose–Einstein condensates (see e.g. [4], [27], [42]).

The article is organized as follows. In Section 2 we introduce the notions of characteristic values and index of an operator with respect to a contour, following mainly [17] and [18]. These tools are used throughout the article; note in particular that they avoid the use of the regularized determinants. Further, we study the asymptotic distribution as $r \searrow 0$ of the characteristic values in a domain of size r, situated at a distance r from the origin (see Theorem 3.1). Then, we prove a general result concerning the asymptotics as $r \searrow 0$ of the characteristic values in a domain of size 1 situated at a distance r from the origin (see Theorem 3.7, Corollary 3.9 and Corollary 3.11). These abstract results are stated in Section 3, and are proved respectively in Section 4 and in Section 5. In Section 6, we apply our abstract results to magnetic Schrödinger operators. Eventually, in Section 7, we construct some counterexamples which show that the assumptions of the results of Section 3 can not be removed.

2. Characteristic values of holomorphic operators

In this section, we define the notions of characteristic values of an operator valued holomorphic function and their multiplicities. For more details, we refer to [18] and to Section 4 of [17].

For the formulation of our results we need the following notations used throughout the article. Let $\mathcal{H}$ be a separable Hilbert space. We denote by $\mathcal{L}(\mathcal{H})$ (resp. $\mathcal{S}_\infty(\mathcal{H})$) the class of linear bounded (resp. compact) operators acting in $\mathcal{H}$. By $\mathcal{GL}(\mathcal{H})$, we denote the class of invertible bounded operators, and by $\mathcal{S}_p(\mathcal{H})$, $p \in [1, +\infty[$, the Schatten–von Neumann classes of compact operators. In particular $\mathcal{S}_1$ is the trace class, and $\mathcal{S}_2$ is the Hilbert–Schmidt class. When appropriate, we omit the explicit indication of the Hilbert space $\mathcal{H}$ where the operators from a given class act.

Definition 2.1. *For $w \in \mathbb{C}$, let $\mathcal{U}$ be a neighborhood of w, and let $F : \mathcal{U} \setminus \{w\} \longrightarrow \mathcal{L}(\mathcal{H})$ be a holomorphic function. We say that F is finite meromorphic at w if the Laurent expansion of F at w has the form*

$$F(z) = \sum_{n=m}^{+\infty} (z-w)^n A_n, \qquad m > -\infty,$$

the operators $A_m, \cdots, A_{-1}$ being of finite rank, if $m < 0$.

If, in addition, A_0 is a Fredholm operator, then F is called Fredholm at w, and the Fredholm index of A_0 is called the Fredholm index of F at w.

Remark 2.2. *Definition 2.1, as well as most of the results of the present section, admits a generalization to a Banach-space setting. We formulate these results in a form sufficient for our purposes.*

Proposition 2.3 ([17, Proposition 4.1.4])**.** *Let $\mathcal{D} \subset \mathbb{C}$ be a connected open set, let $Z \subset \mathcal{D}$ be a discrete and closed subset of $\mathcal{D}$, and let $F : \mathcal{D} \longrightarrow \mathcal{L}(\mathcal{H})$ be a holomorphic function on $\mathcal{D} \setminus Z$. Assume that:*

- *F is finite meromorphic on $\mathcal{D}$, i.e. it is finite meromorphic in a vicinity of each point of Z;*
- *F is Fredholm at each point of $\mathcal{D}$;*
- *there exists $z_0 \in \mathcal{D} \setminus Z$ such that $F(z_0)$ is invertible.*

Then there exists a discrete and closed subset Z' of $\mathcal{D}$ such that:

- $Z \subset Z'$;
- $F(z)$ *is invertible for* $z \in \mathcal{D} \setminus Z'$;
- $F^{-1} : \mathcal{D} \setminus Z' \longrightarrow \mathcal{GL}(\mathcal{H})$ *is finite meromorphic and Fredholm at each point of* $\mathcal{D}$.

Then we can define the *characteristic values* of F, and their multiplicities.

Definition 2.4. *In the setting of Proposition 2.3, each point of Z', where F or F^{-1} is not holomorphic, is called a characteristic value of F. The multiplicity of a characteristic value w_0 is defined by*

$$\operatorname{mult}(w_0) := \frac{1}{2i\pi} \operatorname{tr} \int_{|w-w_0|=\rho} F'(z)F(z)^{-1} dz, \tag{2.1}$$

where $\rho > 0$ is sufficiently small such that $\{w;\ |w - w_0| \leq \rho\} \cap Z' = \{w_0\}$.

By definition, if F is holomorphic in $\mathcal{D}$, a characteristic value of F is a complex number w for which $F(w)$ is not invertible. Then, according to results of [18] and [17, Section 4], $\operatorname{mult}(w)$ is an integer. Moreover, the definition of the multiplicity coincides with a definition of the order of w as a zero of F (see [18] for more details).

If $\Omega \subset \mathcal{D}$ is a connected domain such that $\partial\Omega \cap Z' = \emptyset$, then the sum of the multiplicities of the characteristic values of F inside Ω is the so-called *index of F with respect to the contour* $\partial\Omega$, given by

$$\operatorname{Ind}_{\partial\Omega} F := \frac{1}{2i\pi} \operatorname{tr} \int_{\partial\Omega} F'(z)F(z)^{-1} dz = \frac{1}{2i\pi} \operatorname{tr} \int_{\partial\Omega} F(z)^{-1}F'(z)\, dz. \tag{2.2}$$

We easily check that

$$\operatorname{Ind}_{\partial\Omega}(F_1F_2) = \operatorname{Ind}_{\partial\Omega} F_1 + \operatorname{Ind}_{\partial\Omega} F_2, \tag{2.3}$$

provided that the operator-valued functions F_1 and F_2 satisfy the assumptions of Proposition 2.3. Let us remark also that if $I - F \in \mathcal{S}_1$, then $F'(z)F(z)^{-1} \in \mathcal{S}_1$ for $z \in \mathcal{D} \setminus Z'$, and we have

$$\operatorname{Ind}_{\partial\Omega} F = \frac{1}{2i\pi} \int_{\partial\Omega} \operatorname{tr}\big(F'(z)F(z)^{-1}\big)\, dz = \operatorname{ind}_{\partial\Omega} f, \tag{2.4}$$

where $f(z) = \det(F(z))$ is the Fredholm determinant of $F(z)$ and $\operatorname{ind}_{\partial\Omega} f$ is the standard index of a holomorphic function equal to the number of its zeroes in Ω:

$$\operatorname{ind}_{\partial\Omega} f := \frac{1}{2i\pi} \int_{\partial\Omega} \frac{f'(z)}{f(z)}\, dz. \tag{2.5}$$

More generally, if $I - F \in \mathcal{S}_p$ with integer $p \geq 2$, then the regularized determinant of F is well defined, namely

$$f_p(z) = \det{}_p(F(z)) := \det\Big(F(z) \exp\Big(\sum_{k=1}^{p-1} \frac{1}{k}(I - F(z))^k\Big)\Big),$$

(see [23], [24], [25]), and if $\partial\Omega \cap Z' = \emptyset$, we have

$$\operatorname{ind}_{\partial\Omega} f_p = \frac{1}{2i\pi} \int_{\partial\Omega} \operatorname{tr}\Big(F'(z)F(z)^{-1} - \sum_{k=1}^{p-1} F'(z)(I - F(z))^{k-1}\Big)\, dz = \operatorname{Ind}_{\partial\Omega} F, \tag{2.6}$$

since the sum in the above formula is the derivative of a function.

Moreover, we have a Rouché-type theorem:

Theorem 2.5 ([17, Theorem 4.4.3], [18, Theorem 2.2]). *For $\mathcal{D} \subset \mathbb{C}$ a bounded open set with piecewise C^1-boundary and $Z \subset \mathcal{D}$ a finite set, let $F : \overline{\mathcal{D}} \setminus Z \longrightarrow \mathcal{GL}(\mathcal{H})$ be a holomorphic function which is finite meromorphic and Fredholm at each point of Z and let $G : \overline{\mathcal{D}} \setminus Z \longrightarrow \mathcal{L}(\mathcal{H})$ be a holomorphic function which is finite meromorphic at each point of Z, and satisfies*

$$\|F(z)^{-1}G(z)\| < 1, \qquad z \in \partial\mathcal{D}.$$

Then $F + G$ is finite meromorphic and Fredholm at each point of Z, and

$$\mathrm{Ind}_{\partial\mathcal{D}}(F + G) = \mathrm{Ind}_{\partial\mathcal{D}}(F).$$

3. Asymptotic expansions: abstract results

Let $\mathcal{D}$ be a domain of $\mathbb{C}$ containing 0, and $\mathcal{H}$ be a separable Hilbert space. We consider a holomorphic operator function

$$A : \mathcal{D} \longrightarrow \mathcal{S}_\infty(\mathcal{H}).$$

For $\Omega \subset \mathcal{D}$, we denote by $\mathcal{Z}(\Omega)$ the set of the characteristic values of $I - \frac{A(z)}{z}$, i.e.

$$\mathcal{Z}(\Omega) := \Big\{ z \in \Omega \setminus \{0\};\ I - \frac{A(z)}{z} \text{ is not invertible} \Big\},$$

and by $\mathcal{N}(\Omega)$ the number of characteristic values in Ω counted with their multiplicities, i.e.

$$\mathcal{N}(\Omega) := \#\mathcal{Z}(\Omega).$$

We refer to Section 2 for details concerning the characteristic values.

We deduce from Proposition 2.3 that $\mathcal{Z}(\mathcal{D})$ is a finite set in a neighborhood of the origin as soon as $A(0)$ is of finite rank. In this section we assume that $A(0)$ is a selfadjoint operator and we are mainly interested in the case where $A(0)$ is of infinite rank.

We will formulate results concerning the number of characteristic values of $I - \frac{A(z)}{z}$ in two types of domains: small domains of the form $s\Omega$ with $\Omega \Subset \mathbb{C} \setminus \{0\}$ fixed and s tending to 0, and sectorial domains of the form

$$\mathcal{C}_\theta(a, b) := \{x + iy \in \mathbb{C};\ a \le x \le b,\ |y| \le \theta |x|\}, \tag{3.1}$$

with $b, \theta > 0$ fixed and $a > 0$ tending to 0.

Our goal is to describe situations where the behaviour of $\mathcal{N}(s\Omega)$ as $s \searrow 0$, or of $\mathcal{N}(\mathcal{C}_\theta(r, 1))$ as $r \searrow 0$, is related to the asymptotics of the number

$$n(\Lambda) := \mathrm{tr} \mathbb{1}_\Lambda(A(0)),$$

of the eigenvalues of the operator $A(0)$, lying in an appropriate set $\Lambda \subset \mathbb{R}$, and counted with their multiplicities. Let Π_0 be the orthogonal projection onto $\ker A(0)$, and $\overline{\Pi}_0 := I - \Pi_0$. In small domains we have:

Theorem 3.1. *Let $\mathcal{D}$ be a domain of $\mathbb{C}$ containing 0 and let A be a holomorphic operator-valued function*

$$A : \mathcal{D} \longrightarrow \mathcal{S}_\infty(\mathcal{H}),$$

such that $A(0)$ is selfadjoint and $I - A'(0)\Pi_0$ is invertible. Assume that $\Omega \Subset \mathbb{C} \setminus \{0\}$ is a bounded domain with smooth boundary $\partial\Omega$ which is transverse to the real axis at each point of $\partial\Omega \cap \mathbb{R}$. Then, for all $\delta > 0$ small enough, there exists $s(\delta) > 0$ such that, for all $0 < s < s(\delta)$, we have

$$\mathcal{N}(s\Omega) = n(sJ) + \mathcal{O}\big(n(sI_\delta)|\ln \delta|^2\big),$$

where $J := \Omega \cap \mathbb{R}$, $I_\delta := \partial\Omega \cap \mathbb{R} + [-\delta, \delta]$ and the $\mathcal{O}$ is uniform with respect to s, δ.

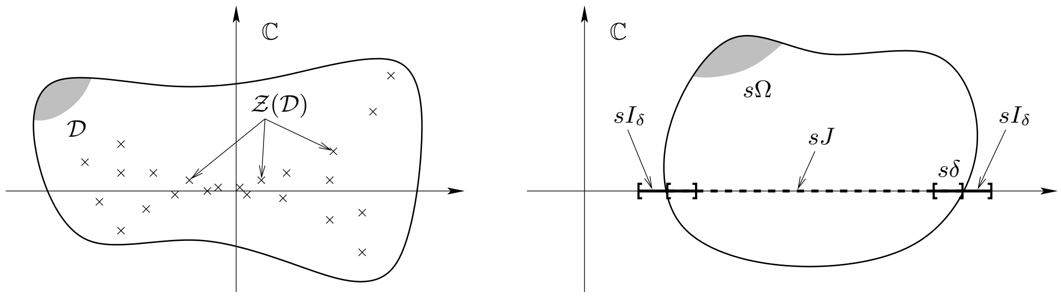

FIGURE 1. The set of characteristic values $\mathcal{Z}(\mathcal{D})$ and the setting of Theorem 3.1.

Remark 3.2. *In the context of Theorem 3.1, the assumptions of Proposition 2.3 hold true and then the characteristic values are well defined. This follows from the hypotheses of Theorem 3.1 and Proposition 3.6 below.*

Remark 3.3. *In Theorem 3.1, the remainder estimate is uniform with respect to some perturbations of the domain Ω. For example, let $\theta > 0$ and $0 < a_- \leq a_+ < b_- \leq b_+ < +\infty$. Then the conclusion of Theorem 3.1 holds for $\Omega = \mathcal{C}_\theta(a, b)$ uniformly with respect to a, b such that $a_- \leq a \leq a_+$ and $b_- \leq b \leq b_+$.*

The setting is illustrated on Figure 1. With respect to different types of Ω, Theorem 3.1 implies the following properties on the characteristic values near 0:

Corollary 3.4. *Under the assumptions of Theorem 3.1, we have*

i) If $\Omega \cap \mathbb{R} = \emptyset$ then $\mathcal{N}(s\Omega) = 0$ for s small enough. This implies that the characteristic values $z \in \mathcal{Z}(\mathcal{D})$ near 0 satisfy

$$|\operatorname{Im} z| = o(|z|).$$

ii) Moreover, if $A(0)$ has a definite sign, i.e. $\pm A(0) \geq 0$, then the characteristic values z near 0 satisfy

$$\pm \operatorname{Re} z \geq 0.$$

iii) If $A(0)$ is of finite rank, then there are no characteristic values in a pointed neighborhood of 0. Moreover, if $A(0)\mathbb{1}_{[0,+\infty[}(\pm A(0))$ is of finite rank, then there are no characteristic values in a neighborhood of 0 intersected with $\{\pm \operatorname{Re} z > 0\}$.

Remark 3.5. *According to Proposition 2.3, the first sentence of Corollary 3.4 iii) holds true even if $A(0)$ is non selfadjoint.*

In fact Corollary 3.4 is a consequence of the following result which, under an appropriate spectral condition on $A(0)$, guarantees the existence of a region free of characteristic values of $I - A(z)/z$. We omit its proof since it is quite similar to the one of Lemma 4.1 below.

Proposition 3.6. *Assume the hypotheses of Theorem 3.1. Let the operator-valued function*

$$z \longmapsto \Big(I - \frac{A(0)}{z}\Big)^{-1},$$

be well defined and uniformly bounded on the set $S \subset \mathcal{D} \setminus \{0\}$. Then, there exist $r_0 > 0$ such that $\mathcal{Z}(S) \cap \{|z| < r_0\} = \emptyset$, and a constant $C > 0$ independent of r_0 and S, such that

$$\sup_{z \in S \cap \{|z| < r_0\}} \Big\| \Big(I - \frac{A(z)}{z}\Big)^{-1} \Big\| \leq C \sup_{z \in S \cap \{|z| < r_0\}} \Big\| \Big(I - \frac{A(0)}{z}\Big)^{-1} \Big\|,$$

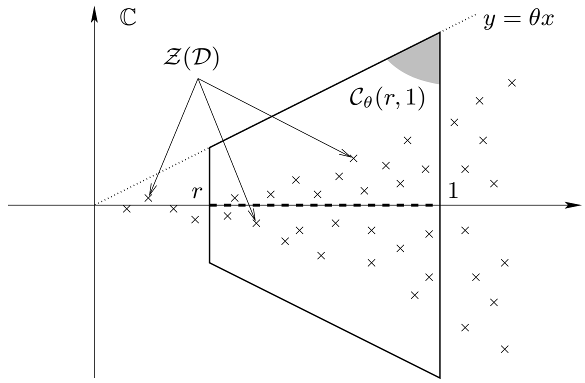

FIGURE 2. The setting of Theorem 3.7 and its corollaries.

if $S \cap \{|z| < r_0\} \neq \emptyset$.

Exploiting Theorem 3.1 with appropriate domains Ω, we obtain the following result in sectorial domains:

Theorem 3.7. *Let* $\mathcal{D}$ *be a domain of* $\mathbb{C}$ *containing* 0 *and let a holomorphic operator function*

$$A : \mathcal{D} \longrightarrow \mathcal{S}_\infty(\mathcal{H}),$$

such that $A(0)$ *is selfadjoint and* $I - A'(0)\Pi_0$ *is invertible. For* $\theta > 0$ *fixed, let* $\mathcal{C}_\theta(r, 1) \subset \mathcal{D}$ *be defined as in* (3.1)*. Then, for all* $\delta > 0$ *small enough, there exists* $s(\delta) > 0$ *such that, for all* $0 < s < s(\delta)$*, we have*

$$\mathcal{N}(\mathcal{C}_\theta(r, 1)) = n([r, 1])\big(1 + \mathcal{O}\big(\delta |\ln \delta|^2\big)\big) + \mathcal{O}\big(|\ln \delta|^2\big) n([r(1 - \delta), r(1 + \delta)]) + \mathcal{O}_\delta(1),$$

where the $\mathcal{O}$*'s are uniform with respect to* s, δ *but the* $\mathcal{O}_\delta$ *may depend on* δ*.*

Remark 3.8. *i) The same result holds mutatis mutandis for* $\mathcal{C}_\theta(-1, -r)$*,* $r > 0$*, under the same assumptions. To prove this point, it is sufficient to change the variable* $z \to -z$ *and replace* $A(z)$ *by* $-A(-z)$*.*

ii) The dependence on θ *in Theorem 3.7 is inessential. More precisely, thanks to Corollary 3.4 i), the difference* $\mathcal{N}(\mathcal{C}_\theta(r, 1)) - \mathcal{N}(\mathcal{C}_\nu(r, 1))$ *is constant for* r *small enough.*

Now, let us give conditions on $n([r, 1])$ for which Theorem 3.7 implies asymptotic expansion of $\mathcal{N}(\mathcal{C}_\theta(r, 1))$ with main term $n([r, 1])$.

Corollary 3.9. *Let the assumptions of Theorem 3.7 hold true. Suppose that there exists* $\gamma > 0$ *such that*

$$n([r, 1]) = \mathcal{O}(r^{-\gamma}), \qquad r \searrow 0, \tag{3.2}$$

and that $n([r, 1])$ *grows unboundedly as* $r \searrow 0$*. Then there exists a positive sequence* $(r_k)_{k \in \mathbb{N}}$ *tending to* 0*, which satisfies*

$$\mathcal{N}(\mathcal{C}_\theta(r_k, 1)) = n([r_k, 1])(1 + o(1)), \qquad k \to +\infty. \tag{3.3}$$

Remark 3.10. *Note that* (3.2) *is equivalent to the fact that* $A(0)\mathbb{1}_{[0,+\infty[}(A(0)) \in \mathcal{S}_p$ *for some* $p \in [1, +\infty[$*. Such hypothesis is reasonable since it guarantees that, in some sense,* $A(0)$ *has less eigenvalues in a vicinity of* r *than in the whole interval* $[r, 1]$ *(see Lemma 5.2 below). On*

the other hand, the assumption that $n([r,1])$ *grows unboundedly as* $r \searrow 0$ *is equivalent to* $\operatorname{rank} A(0)\mathbb{1}_{[0,+\infty[}(A(0)) = +\infty$.

Corollary 3.11. *Let the assumptions of Theorem 3.7 hold true. Suppose that*

$$n([r,1]) = \Phi(r)(1+o(1)), \qquad r \searrow 0,$$

with $\Phi(r) = Cr^{-\gamma}$, *or* $\Phi(r) = C|\ln r|^{\gamma}$, *or* $\Phi(r) = C\frac{|\ln r|}{\ln|\ln r|}$, *for some* $\gamma, C > 0$. *Then*

$$\mathcal{N}(\mathcal{C}_\theta(r,1)) = \Phi(r)(1+o(1)), \qquad r \searrow 0.$$

These results are proved in Section 5.

Remark 3.12. *As shown in Section 7, the assumptions of Theorem 3.1 and Theorem 3.7 (compactness of* $A(z)$*, selfadjointness of* $A(0)$*, and invertibility of* $I - A'(0)\Pi_0$*) are necessary in order to have the claimed results.*

4. Proof of Theorem 3.1

Throughout the section we assume that the hypotheses of Theorem 3.1 are fulfilled. Note that it is enough to prove this result for δ small enough. In the following, C will denote a positive constant independent of δ, s, z which may change its value from line to line.

4.1. **Estimate of the resolvents.**

For $0 < \delta \le \delta_0 := \min(1, \operatorname{dist}(\partial\Omega \cap \mathbb{R}, \{0\})/2)$ and $s > 0$, let us introduce the finite rank operator

$$K_\delta(s) := A(0)\mathbb{1}_{I_\delta}\Big(\frac{A(0)}{s}\Big), \tag{4.1}$$

where $I_\delta = \partial\Omega \cap \mathbb{R} + [-\delta, \delta]$. This operator will be used to "remove" the eigenvalues of $A(0)$ which are at distance δs from $\partial\Omega \cap \mathbb{R}$. We also define the set

$$V_{\nu,\delta} := (\Omega + B(0,\nu)) \cap \{|\operatorname{Im} z| > \varepsilon\delta\} \qquad \text{and} \qquad W_\delta := (\partial\Omega \cap \mathbb{R}) + B(0,\delta),$$

where $\nu > 0$ is a constant chosen sufficiently small so that $\Omega + B(0,\nu) \Subset \mathbb{C}\setminus\{0\}$ and $\varepsilon \in]0, 1/2[$ will be chosen later so that $\partial\Omega \subset V_{\nu/2,2\delta} \cup W_{\delta/4}$. For $\delta > 0$ small enough, the setting is illustrated in Figure 3. First, we obtain estimates on the free and perturbed resolvents which hold uniformly with respect to δ.

Lemma 4.1. *There exists* $C > 0$ *such that, for all* $0 < \delta \le \delta_0$*, there exists* $s(\delta) > 0$ *such that, for all* $0 < s < s(\delta)$*, we have*

$$\Big\|\Big(I - \frac{A(sz) - K_\delta(s)}{sz}\Big)^{-1}\Big\| < \frac{C}{\delta + |\operatorname{Im} z|}, \tag{4.2}$$

uniformly for $z \in V_{\nu,\delta} \cup W_{\delta/2}$ *and*

$$\Big\|\Big(I - \frac{A(sz)}{sz}\Big)^{-1}\Big\| < \frac{C}{\delta + |\operatorname{Im} z|}, \tag{4.3}$$

uniformly for $z \in V_{\nu,\delta}$.

Proof. We begin by proving (4.2). First, we make the following decomposition

$$I - \frac{A(sz) - K_\delta(s)}{sz} = \Big(I - \frac{A(sz) - A(0)}{sz}\Big(I - \frac{A(0) - K_\delta(s)}{sz}\Big)^{-1}\Big)\Big(I - \frac{A(0) - K_\delta(s)}{sz}\Big). \tag{4.4}$$

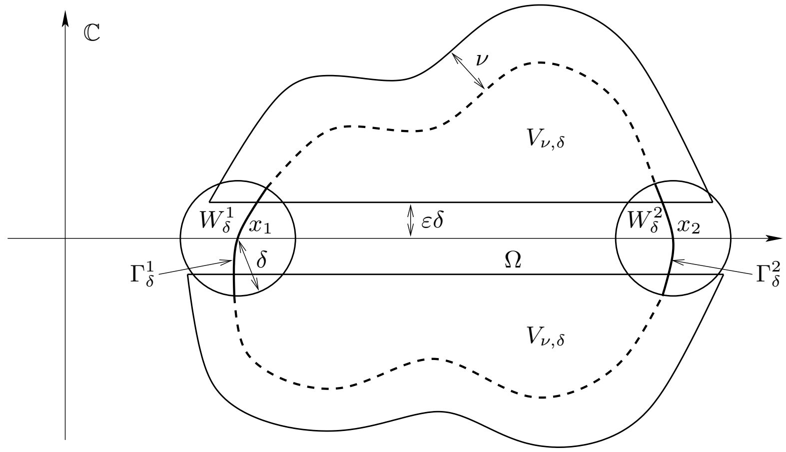

FIGURE 3. The domains $V_{\nu,\delta}$ and $W_\delta = W_\delta^1 \cup W_\delta^2$, the set $\partial\Omega \cap \mathbb{R} = \{x_1, x_2\}$ and the paths $\Gamma_\delta = \Gamma_\delta^1 \cup \Gamma_\delta^2$ with $\Gamma_\delta^j = \partial\Omega \cap B(x_j, \delta)$.

From the definition (4.1) of $K_\delta(s)$ and the spectral theorem, we get

$$\Big\| \Big(I - \frac{A(0) - K_\delta(s)}{sz} \Big)^{-1} \Big\| \leq \frac{C}{\delta + |\operatorname{Im} z|}, \tag{4.5}$$

uniformly for $0 < \delta \leq \delta_0$, $s \in]0,1]$ and $z \in V_{\nu,\delta} \cup W_{\delta/2}$.

We will now prove that, for δ fixed,

$$\operatorname*{s-lim}_{s \to 0} \Big(\overline{\Pi}_0 \Big(I - \frac{A(0) - K_\delta(s)}{sz} \Big)^{-1} \Big)^* = 0, \tag{4.6}$$

uniformly for $z \in V_{\nu,\delta} \cup W_{\delta/2}$. For $\alpha > 0$ fixed, according to the spectral theorem, there exists $M > 0$ such that

$$\Big\| \mathbb{1}_{\{|\lambda| \geq Ms\}}(A(0)) \Big(I - \frac{A(0) - K_\delta(s)}{sz} \Big)^{-1} \Big\| < \alpha,$$

uniformly with respect to $s \in]0,1]$ and $z \in V_{\nu,\delta} \cup W_{\delta/2}$. On the other hand, the projection $\mathbb{1}_{\{0<|\lambda|<Ms\}}(A(0))$ tends strongly to 0 as s tends to 0. Then, using (4.5), we deduce that, for δ fixed,

$$\operatorname*{s-lim}_{s \to 0} \Big(\mathbb{1}_{\{0<|\lambda|<Ms\}}(A(0)) \Big(I - \frac{A(0) - K_\delta(s)}{sz} \Big)^{-1} \Big)^* = 0,$$

uniformly with respect to $z \in V_{\nu,\delta} \cup W_{\delta/2}$. The two last equations imply (4.6).

Since A is holomorphic near 0, there exists a holomorphic operator-valued function R_2 such that

$$\frac{A(sz) - A(0)}{sz} = A'(0) + szR_2(sz).$$

Then

$$\begin{aligned} I - \frac{A(sz) - A(0)}{sz} &\Big(I - \frac{A(0) - K_\delta(s)}{sz} \Big)^{-1} \\ &= I - A'(0)\Pi_0 - \big(A'(0)\overline{\Pi}_0 + szR_2(sz) \big) \Big(I - \frac{A(0) - K_\delta(s)}{sz} \Big)^{-1}. \end{aligned} \tag{4.7}$$

Exploiting (4.5), (4.6), $szR_2(sz) = \mathcal{O}(s)$, and the compactness of $A'(0)$, we deduce that, for δ fixed, the norm of the last term at the right hand side of (4.7) tends to 0 as $s \to 0$, uniformly

with respect to $z \in V_{\nu,\delta} \cup W_{\delta/2}$. At last, $I - A'(0)\Pi_0$ is invertible by assumption. Then there exists $C > 0$ such that, for all $0 < \delta \leq \delta_0$, we can choose $s(\delta) > 0$ sufficiently small such that

$$\Big\| \Big(I - \frac{A(sz) - A(0)}{sz} \Big(I - \frac{A(0) - K_\delta(s)}{sz} \Big)^{-1} \Big)^{-1} \Big\| < C, \tag{4.8}$$

uniformly for $0 < s < s(\delta)$ and $z \in V_{\nu,\delta} \cup W_{\delta/2}$. Then, (4.2) follows from (4.4), (4.5) and (4.8).

The proof of (4.3) is similar. Instead of (4.5), we use

$$\Big\| \Big(I - \frac{A(0)}{sz} \Big)^{-1} \Big\| \leq \frac{s|z|}{s|\operatorname{Im} z|} \leq \frac{1}{\delta + |\operatorname{Im} z|},$$

uniformly for $0 < \delta \leq \delta_0$, $s \in]0, 1]$ and $z \in V_{\nu,\delta}$. □

4.2. **Reduction of the problem.**

By the definition of the multiplicity of the characteristic values (2.1) and of the index of an operator (2.2), we have the following

Lemma 4.2. *Assume that $s\Omega \subset \mathcal{D}$ and that there are no characteristic values of $I - A(z)/z$ on $s\partial\Omega$. Then,*

$$\mathcal{N}(s\Omega) = \operatorname{Ind}_{\partial\Omega} \Big(I - \frac{A(sz)}{sz} \Big).$$

Note that (4.3) implies Corollary 3.4 *i*). Therefore, since the characteristic values form a discrete set, the assumptions of Lemma 4.2 are satisfied for almost all s small enough. Moreover, from the statement of Theorem 3.1, it is enough to prove it for almost all s small enough. Then, we may always assume in the sequel that the assumptions of Lemma 4.2 are satisfied.

Now, we define

$$\begin{aligned} f_\delta(s, z) &= \det \Big(\Big(I - \frac{A(sz)}{sz} \Big) \Big(I - \frac{A(sz) - K_\delta(s)}{sz} \Big)^{-1} \Big) \\ &= \det \Big(I - \frac{K_\delta(s)}{sz} \Big(I - \frac{A(sz) - K_\delta(s)}{sz} \Big)^{-1} \Big), \end{aligned} \tag{4.9}$$

the relative determinant which is well defined for $z \in V_{\nu,\delta} \cup W_{\delta/2}$ by Lemma 4.1 and the finiteness of $\operatorname{rank} K_\delta(s)$.

Lemma 4.3. *For all $0 < \delta \leq \delta_0$, there exists $s(\delta) > 0$ such that, for all $0 < s < s(\delta)$,*

$$\operatorname{Ind}_{\partial\Omega} \Big(I - \frac{A(sz)}{sz} \Big) = \operatorname{Ind}_{\partial\Omega} \Big(I - \frac{A(sz) - K_\delta(s)}{sz} \Big) + \operatorname{ind}_{\partial\Omega} f_\delta(s, z),$$

where the index of a holomorphic function is defined in (2.5).

Proof. Remark that all the quantities are well defined on $\partial\Omega$ since we have assumed the hypotheses of Lemma 4.2. We have

$$I - \frac{A(sz)}{sz} = I - \frac{A(sz) - K_\delta(s)}{sz} - \frac{K_\delta(s)}{sz},$$

and we can then write

$$I - \frac{A(sz)}{sz} = \Big(I - \frac{K_\delta(s)}{sz} \Big(I - \frac{A(sz) - K_\delta(s)}{sz} \Big)^{-1} \Big) \Big(I - \frac{A(sz) - K_\delta(s)}{sz} \Big).$$

Thus we deduce the lemma from (2.3)–(2.4). □

Lemma 4.4. *For all $0<\delta\le\delta_0$, there exists $s(\delta)>0$ such that, for all $0<s<s(\delta)$,*

$$\operatorname{Ind}_{\partial\Omega}\Big(I-\frac{A(sz)-K_\delta(s)}{sz}\Big)=\operatorname{Ind}_{\partial\Omega}\Big(I-\frac{A(0)-K_\delta(s)}{sz}\Big).$$

Proof. First, by using the following decomposition

$$I-\frac{A(sz)-K_\delta(s)}{sz}=\Big(I-\frac{A(sz)-A(0)}{sz}\Big(I-\frac{A(0)-K_\delta(s)}{sz}\Big)^{-1}\Big)\Big(I-\frac{A(0)-K_\delta(s)}{sz}\Big),$$

we have

$$\begin{aligned}\operatorname{Ind}_{\partial\Omega}\Big(I-\frac{A(sz)-K_\delta(s)}{sz}\Big)&=\operatorname{Ind}_{\partial\Omega}\Big(I-\frac{A(0)-K_\delta(s)}{sz}\Big)\\&\quad+\operatorname{Ind}_{\partial\Omega}\Big(I-\frac{A(sz)-A(0)}{sz}\Big(I-\frac{A(0)-K_\delta(s)}{sz}\Big)^{-1}\Big).\end{aligned}\tag{4.10}$$

From (4.7), we can write

$$\begin{aligned}I-&\frac{A(sz)-A(0)}{sz}\Big(I-\frac{A(0)-K_\delta(s)}{sz}\Big)^{-1}\\&=I-A'(0)\Pi_0-\big(A'(0)\overline{\Pi}_0+szR_2(sz)\big)\Big(I-\frac{A(0)-K_\delta(s)}{sz}\Big)^{-1}.\end{aligned}$$

Moreover the discussion below (4.7) shows that the last term of the above equality tends to 0 as s tends to 0 uniformly for $z\in\partial\Omega$ with $0<\delta\le\delta_0$ fixed. Then, since $I-A'(0)\Pi_0$ is invertible, using the Rouché theorem (see Theorem 2.5), we deduce

$$\operatorname{Ind}_{\partial\Omega}\Big(I-\frac{A(sz)-A(0)}{sz}\Big(I-\frac{A(0)-K_\delta(s)}{sz}\Big)^{-1}\Big)=\operatorname{Ind}_{\partial\Omega}\big(I-A'(0)\Pi_0\big)=0.$$

Combining with (4.10), this concludes the proof. $\square$

Proposition 4.5. *Let us consider the function f_δ introduced in (4.9). For all $\delta>0$ small enough, there exists $s(\delta)>0$ such that, for all $0<s<s(\delta)$,*

$$\operatorname{ind}_{\partial\Omega}f_\delta(s,\cdot)=\mathcal{O}\big(n(sI_\delta)|\ln\delta|^2\big).$$

We now prove Theorem 3.1 and postpone the proof of the crucial Proposition 4.5 to the next section. Combining Lemma 4.2, Lemma 4.3, Lemma 4.4 with Proposition 4.5, we obtain

$$\mathcal{N}(s\Omega)=\operatorname{Ind}_{\partial\Omega}\Big(I-\frac{A(0)-K_\delta(s)}{sz}\Big)+\mathcal{O}\big(n(sI_\delta)|\ln\delta|^2\big).$$

Let $\widetilde{A}_0:=A(0)-K_\delta(s)=A(0)\mathbb{1}_{\mathbb{R}\setminus sI_\delta}(A(0))$ and $\widetilde{I}_\delta:=I_\delta\cap\Omega$. Then Theorem 3.1 follows from

$$\begin{aligned}\operatorname{Ind}_{\partial\Omega}\Big(I-\frac{A(0)-K_\delta(s)}{sz}\Big)&=\frac{1}{2i\pi}\operatorname{tr}\int_{\partial\Omega}\frac{\widetilde{A}_0}{sz}\Big(z-\frac{\widetilde{A}_0}{s}\Big)^{-1}dz=\frac{1}{2i\pi}\operatorname{tr}\int_{\partial\Omega}\Big(z-\frac{\widetilde{A}_0}{s}\Big)^{-1}dz\\&=\operatorname{tr}\mathbb{1}_{sJ}(\widetilde{A}_0)=\operatorname{tr}\mathbb{1}_{sJ\setminus s\widetilde{I}_\delta}(A(0))=n(sJ)-n(s\widetilde{I}_\delta)\\&=n(sJ)+\mathcal{O}\big(n(sI_\delta)\big).\end{aligned}$$

In the same way, Remark 3.3 follows from the fact that Proposition 4.5 holds uniformly with respect to $a_-\le a\le a_+$ and $b_-\le b\le b_+$.

4.3. **Proof of Proposition 4.5.**

We first obtain a factorization of the determinant $f_\delta(s,z)$. This idea, due to J. Sjöstrand [40], comes from the study of the semiclassical resonances. Since $\partial\Omega$ is transverse to the real axis, the intersection $\partial\Omega \cap \mathbb{R}$ is a finite set of real numbers x_j with $j = 1, \dots, J$. Setting $W_\delta^j := B(x_j, \delta)$, we have

$$W_\delta = \bigcup_{j=1}^{J} W_\delta^j.$$

Note that this union is disjoint for δ small enough (see Figure 3).

Lemma 4.6. *There exists $C > 0$ such that, for all $\delta > 0$ small enough, there exists $s(\delta) > 0$ such that, for all $0 < s < s(\delta)$,*

i) for all j, the function $f_\delta(s,z)$ can be written in $W_{\delta/4}^j$ as

$$f_\delta(s,z) = \prod_{k=1}^{N_\delta(s)} \frac{(z - z_k^\delta(s))}{\delta} e^{g_\delta(s,z)},$$

where $z_k^\delta(s) \in W_{\delta/2}^j$, $z \longmapsto g_\delta(s,z)$ is holomorphic in $W_{\delta/4}^j$ with

$$N_\delta(s) \le C|\ln\delta| n(sI_\delta), \tag{4.11}$$

and

$$\left|g_\delta'(s,z)\right| \le C\frac{|\ln\delta|}{\delta} n(sI_\delta), \qquad z \in W_{\delta/4}^j.$$

ii) the function $f_\delta(s,z)$ has no zeroes in $V_{\nu/2,2\delta}$ and, in this set,

$$f_\delta(s,z) = e^{g_\delta(s,z)}$$

with $z \longmapsto g_\delta(s,z)$ holomorphic in $V_{\nu/2,2\delta}$ and satisfying $\left|g_\delta'(s,z)\right| \le C\frac{\langle \ln|\operatorname{Im} z|\rangle}{|\operatorname{Im} z|} n(sI_\delta)$.

To prove Lemma 4.6 *i*), we will apply a complex-analysis result obtained by J. Sjöstrand in the context of the investigation of resonance distribution:

Theorem 4.7 (see [40], [41])**.** *Let U be a simply connected domain of $\mathbb{C}$ with $U \cap \{\operatorname{Im}\lambda > 0\} \neq \emptyset$. Let $F : U \longrightarrow \mathbb{C}$ be a holomorphic function such that for some $M \ge 1$ we have*

$$\begin{aligned} |F(\lambda)| &\le e^{M}, && \lambda \in U, \\ |F(\lambda)| &\ge e^{-M}, && \lambda \in U \cap \{\operatorname{Im}\lambda > 0\}. \end{aligned}$$

Then, for any $\widetilde{U} \Subset U$, there exists a constant $C_{\widetilde{U},U}$ independent of F and M, and a holomorphic function $g : \widetilde{U} \longrightarrow \mathbb{C}$ such that

$$F(\lambda) = \prod_{k=1}^{N} (\lambda - \lambda_k) e^{g(\lambda)}, \qquad \lambda \in \widetilde{U},$$

where the λ_k's are zeroes of F in U, $N \le C_{\widetilde{U},U} M$, and $|g'(\lambda)| \le C_{\widetilde{U},U} M$ for $\lambda \in \widetilde{U}$.

Proof of Lemma 4.6. Since $\mathcal{K}(z) := \frac{K_\delta(s)}{sz}\Big(I - \frac{A(sz)-K_\delta(s)}{sz}\Big)^{-1}$ is a finite rank operator, we have

$$f_\delta(s,z) = \det\big(I - \mathcal{K}(z)\big) = \prod_{j=1}^{\operatorname{rank}\mathcal{K}(z)} (1-\lambda_j(z)),$$

where $\lambda_j(z)$ are the eigenvalues of $\mathcal{K}(z)$. On the other hand, the definition of the operator $K_\delta(s) = A(0)\mathbb{1}_{I_\delta}(\frac{A(0)}{s})$ in (4.1), and Lemma 4.1 yield

$$\operatorname{rank}\mathcal{K}(z) = n(sI_\delta), \qquad \|\mathcal{K}(z)\| < \frac{C}{\delta + |\operatorname{Im} z|}, \qquad z \in V_{\nu,\delta} \cup W_{\delta/2}.$$

Then, there exists $C>0$ such that, for all $\delta > 0$ small enough and then s small enough,

$$\begin{aligned} |f_\delta(s,z)| &\leq \big(1 + \|\mathcal{K}(z)\|\big)^{\operatorname{rank}\mathcal{K}(z)} \\ &\leq e^{Cn(sI_\delta)|\ln(\delta+|\operatorname{Im} z|)|}, \end{aligned} \tag{4.12}$$

uniformly for $z \in V_{\nu,\delta} \cup W_{\delta/2}$. On the other hand, for $z \in V_{\nu,\delta}$,

$$f_\delta(s,z)^{-1} = \det\Big(I + \frac{K_\delta(s)}{sz}\Big(I - \frac{A(sz)}{sz}\Big)^{-1}\Big).$$

Applying once again Lemma 4.1, the previous argument gives

$$|f_\delta(s,z)| \geq e^{-Cn(sI_\delta)|\ln(\delta+|\operatorname{Im} z|)|}, \tag{4.13}$$

for $z \in V_{\nu,\delta}$.

Now we apply Theorem 4.7 to the function $\lambda \longmapsto F(\lambda,s,\delta) := f_\delta(s,x_j+\delta\lambda)$ which is holomorphic in $B(0,1/2)$. Estimates (4.12)–(4.13) give

$$\begin{aligned} |F(\lambda,s,\delta)| &\leq e^{Cn(sI_\delta)|\ln\delta|}, && \lambda \in B(0,1/2), \\ |F(\lambda,s,\delta)| &\geq e^{-Cn(sI_\delta)|\ln\delta|}, && \lambda \in B(0,1/2) \cap \{\operatorname{Im} z > \varepsilon\}. \end{aligned}$$

Then, Theorem 4.7 yields, for all $\lambda \in B(0,1/4)$,

$$f_\delta(s,x_j+\delta\lambda) = \prod_{k=1}^{N_\delta(s)} (\lambda - \lambda_k^\delta(s)) e^{g_\delta(s,\lambda)},$$

with $\lambda_k^\delta(s) \in B(0,1/2)$, $N_\delta(s) \leq \widetilde{C}n(sI_\delta)|\ln\delta|$ and $|g_\delta'(s,\lambda)| \leq \widetilde{C}n(sI_\delta)|\ln\delta|$. Changing of variable $z = x_j + \delta\lambda$, we obtain Lemma 4.6 *i)*.

According to (4.13), for sufficiently small s, $f_\delta(s,z)$ has no zeroes in $V_{\nu,\delta}$. Then there exists $g_\delta(s,z)$ holomorphic with respect to $z \in V_{\nu,\delta}$ such that

$$f_\delta(s,z) = e^{g_\delta(s,z)}.$$

For $z \in V_{\nu/2,2\delta}$ consider the function

$$F : \lambda \longmapsto f_\delta\Big(s, z + \lambda\frac{|\operatorname{Im} z|}{4}\Big), \qquad \lambda \in B(0,2).$$

Since F has no zeroes in $B(0,2)$, the combination of (4.12)–(4.13) with Theorem 4.7, yields

$$\frac{|\operatorname{Im} z|}{4}\Big|g_\delta'\Big(s, z + \lambda\frac{|\operatorname{Im} z|}{4}\Big)\Big| \leq \widetilde{C}n(sI_\delta)|\ln|\operatorname{Im} z||, \qquad \lambda \in B(0,1),$$

with $\widetilde{C}$ independent of λ, z, s and δ. Thus we obtain part *ii)* of Lemma 4.6. $\square$

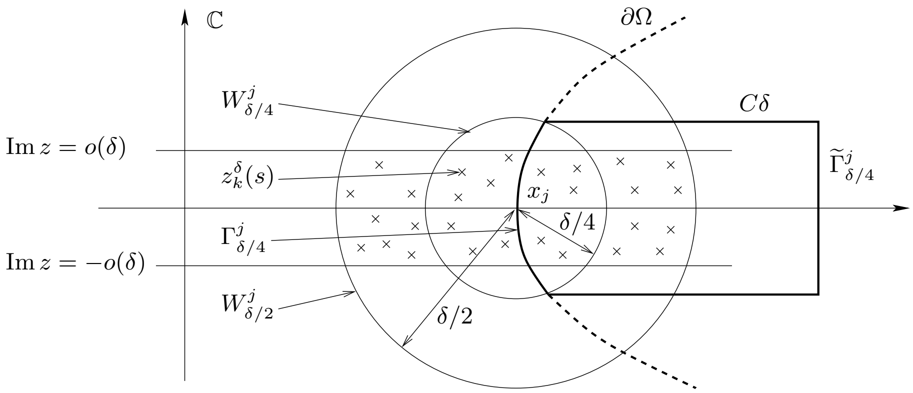

FIGURE 4. The situation near x_j and the set $\widetilde{\Gamma}^j_\delta$.

We can now prove Proposition 4.5. For $j = 1, \dots, J$, set

$$\Gamma^j_\delta := \partial\Omega \cap B(x_j, \delta) \qquad \text{and} \qquad \Gamma_\delta := \bigcup_{j=1}^J \Gamma^j_\delta.$$

For δ small enough, the Γ^j_δ's are segments of size δ (see Figure 3). Moreover, from the assumptions on Ω, there exists $\varepsilon > 0$ such that $\operatorname{Im} z > 2\varepsilon\delta$ for all $z \in \partial\Omega \setminus \Gamma_{\delta/4}$. Let us write

$$\begin{aligned} \operatorname{ind}_{\partial\Omega} f_\delta(s,z) &= \frac{1}{2i\pi} \int_{\partial\Omega} \frac{f'_\delta(s,z)}{f_\delta(s,z)} dz \\ &= \frac{1}{2i\pi} \int_{\partial\Omega \setminus \Gamma_{\delta/4}} \frac{f'_\delta(s,z)}{f_\delta(s,z)} dz + \frac{1}{2i\pi} \int_{\Gamma_{\delta/4}} \frac{f'_\delta(s,z)}{f_\delta(s,z)} dz. \end{aligned} \tag{4.14}$$

From Lemma 4.6 *ii*), the first term of (4.14) satisfies

$$\begin{aligned} \Big| \frac{1}{2i\pi} \int_{\partial\Omega \setminus \Gamma_{\delta/4}} \frac{f'_\delta(s,z)}{f_\delta(s,z)} dz \Big| &\leq \frac{1}{2\pi} \int_{\partial\Omega \setminus \Gamma_{\delta/4}} \big| g'_\delta(s,z) \big| |dz| \\ &\leq \frac{C}{2\pi} n(sI_\delta) \int_{\partial\Omega \setminus \Gamma_{\delta/4}} \frac{\langle \ln |\operatorname{Im} z| \rangle}{|\operatorname{Im} z|} |dz| \\ &\leq C n(sI_\delta) |\ln \delta|^2. \end{aligned} \tag{4.15}$$

Here, we have used the elementary identity $\int_{[\delta,1]} \frac{|\ln x|}{x} \, dx = \frac{1}{2} |\ln \delta|^2$.

On the other hand, Lemma 4.6 *i*) implies

$$\frac{1}{2i\pi} \int_{\Gamma^j_{\delta/4}} \frac{f'_\delta(s,z)}{f_\delta(s,z)} dz = \sum_{k=1}^{N_\delta(s)} \frac{1}{2\pi} \int_{\Gamma^j_{\delta/4}} \frac{1}{z - z^\delta_k(s)} \, dz + \frac{1}{2\pi} \int_{\Gamma^j_{\delta/4}} g'_\delta(s,z) \, dz. \tag{4.16}$$

Since $|\Gamma^j_{\delta/4}| \leq C\delta$, we get

$$\Big| \frac{1}{2\pi} \int_{\Gamma^j_{\delta/4}} g'_\delta(s,z) \, dz \Big| \leq \frac{C}{2\pi} n(sI_\delta) \int_{\Gamma^j_{\delta/4}} \frac{|\ln \delta|}{\delta} |dz| \leq C n(sI_\delta) |\ln \delta|. \tag{4.17}$$

Now, let $\widetilde{\Gamma}^j_{\delta/4}$ be as in Figure 4. Lemma 4.6 yields $z_k^\delta(s) \in B(x_j, \delta/2)$. Moreover, thanks to (4.3), we also have $\operatorname{Im} z_k^\delta(s) = o(\delta)$ as s tends to 0. Therefore, the $z_k^\delta(s)$'s are at distance δ from $\widetilde{\Gamma}^j_{\delta/4}$ for s small enough. Then,

$$\begin{aligned}\Big| \frac{1}{2\pi} \int_{\Gamma^j_{\delta/4}} \frac{1}{z - z_k^\delta(s)}\, dz \Big| &= \Big| \frac{1}{2\pi} \int_{\Gamma^j_{\delta/4} \cup \widetilde{\Gamma}^j_{\delta/4}} \frac{1}{z - z_k^\delta(s)}\, dz \Big| + \Big| \frac{1}{2\pi} \int_{\widetilde{\Gamma}^j_{\delta/4}} \frac{1}{z - z_k^\delta(s)}\, dz \Big| \\ &\leq 1 + \int_{\widetilde{\Gamma}^j_{\delta/4}} \frac{C}{\delta} |dz| \leq C. \end{aligned} \tag{4.18}$$

Combining (4.18) with (4.11), (4.16), and (4.17), we get

$$\Big| \frac{1}{2i\pi} \int_{\Gamma_{\delta/4}} \frac{f_\delta'(s,z)}{f_\delta(s,z)} dz \Big| \leq C n(s I_\delta) |\ln \delta|. \tag{4.19}$$

Proposition 4.5 now follows from (4.14), (4.15), and (4.19).

5. Proof of Theorem 3.7 and its corollaries

In this section we prove Theorem 3.7 applying Theorem 3.1 with appropriate domains $s\Omega$ constructed so that the associated intervals I_δ contain a "small" number of eigenvalues of $A(0)$. First we choose these intervals I_δ in accordance with the following general result for counting functions.

Lemma 5.1. *There exists $C > 0$ such that, for any $\delta > 0$ small enough and $j \in \mathbb{N}$, there exists $\beta_j \in [j - \frac{1}{4}, j + \frac{1}{4}]$ such that*

$$n\big(\big[2^{-\beta_j}(1-\delta), 2^{-\beta_j}(1+\delta) \big] \big) \leq C\delta\, n\big(\big[2^{-(j+\frac{1}{4})}, 2^{-(j-\frac{1}{4})} \big] \big).$$

Proof. First, there exist $\delta_1, \varepsilon_1 > 0$ such that, for all $0 < \delta < \delta_1$, one can find disjoint intervals

$$I_k(\delta) \subset [2^{-\frac{1}{4}}, 2^{\frac{1}{4}}],$$

of the form $[2^{-\beta}(1-\delta), 2^{-\beta}(1+\delta)]$ for some $\beta \in [-\frac{1}{4}, \frac{1}{4}]$, with integer $k \in [0, \varepsilon_1/\delta]$.

Now assume that the assertion of the lemma is not true. Then, for all $C > 0$ and $\delta_0 > 0$, there exists $0 < \delta < \delta_0$ and $j \in \mathbb{N}$ such that, for all $\beta \in [j - \frac{1}{4}, j + \frac{1}{4}]$,

$$n\big(\big[2^{-\beta}(1-\delta), 2^{-\beta}(1+\delta) \big] \big) > C\delta\, n\big(\big[2^{-(j+\frac{1}{4})}, 2^{-(j-\frac{1}{4})} \big] \big).$$

We choose $C = 2/\varepsilon_1$ and $\delta_0 = \delta_1$. Using the intervals $I_k(\delta)$ constructed previously, we get

$$\begin{aligned} n\big(\big[2^{-(j+\frac{1}{4})}, 2^{-(j-\frac{1}{4})} \big] \big) &\geq n\Big(\bigcup_{k=0}^{\varepsilon_1/\delta} 2^{-j} I_k(\delta) \Big) = \sum_{k=0}^{\varepsilon_1/\delta} n\big(2^{-j} I_k(\delta) \big) \\ &> \frac{\varepsilon_1}{\delta} C\delta\, n\big(\big[2^{-(j+\frac{1}{4})}, 2^{-(j-\frac{1}{4})} \big] \big) = 2\, n\big(\big[2^{-(j+\frac{1}{4})}, 2^{-(j-\frac{1}{4})} \big] \big), \end{aligned} \tag{5.1}$$

which gives a contradiction. □

Combining this lemma with Theorem 3.1, we prove Theorem 3.7.

Proof of Theorem 3.7. We consider the sequence $(\beta_j)_j$ constructed in Lemma 5.1. For $\delta > 0$ small enough, let $r(\delta) > 0$ be such that Theorem 3.1 and Remark 3.3 with $a_- = a_+ = 1$, $b_- = 2^{\frac{1}{2}}$ and $b_+ = 2^{\frac{9}{4}}$ hold true for $0 < r < r(\delta)$. In the sequel, $M(\delta)$ will denote the smallest

integer for which $2^{-\beta_{M(\delta)}} < r(\delta)$, and $N(r)$, $0 < r < r(\delta)$, will denote the unique integer such that $2r < 2^{-N(r)} \leq 4r$.

By the disjoint decomposition

$$\mathcal{C}_\theta(r,1) = \mathcal{C}_\theta\big(r, 2^{-\beta_{N(r)}}\big) \bigcup \bigcup_{j=M(\delta)}^{N(r)-1} \mathcal{C}_\theta(2^{-\beta_{j+1}}, 2^{-\beta_j}) \bigcup \mathcal{C}_\theta\big(2^{-\beta_{M(\delta)}}, 1\big),$$

we have

$$\mathcal{N}(\mathcal{C}_\theta(r,1)) = \mathcal{N}\big(\mathcal{C}_\theta\big(r, 2^{-\beta_{N(r)}}\big)\big) + \sum_{j=M(\delta)}^{N(r)-1} \mathcal{N}\big(\mathcal{C}_\theta\big(2^{-\beta_{j+1}}, 2^{-\beta_j}\big)\big) + \mathcal{N}\big(\mathcal{C}_\theta\big(2^{-\beta_{M(\delta)}}, 1\big)\big). \tag{5.2}$$

By construction of β_j, we have $2^{-\beta_{N(r)}} \in r]2^{\frac{3}{4}}, 2^{\frac{9}{4}}]$ and

$$\mathcal{C}_\theta\big(2^{-\beta_{j+1}}, 2^{-\beta_j}\big) = 2^{-\beta_{j+1}} \mathcal{C}_\theta\big(1, 2^{\beta_{j+1}-\beta_j}\big), \qquad \mathcal{C}_\theta\big(1, 2^{\frac{1}{2}}\big) \subset \mathcal{C}_\theta\big(1, 2^{\beta_{j+1}-\beta_j}\big) \subset \mathcal{C}_\theta\big(1, 2^{\frac{3}{2}}\big).$$

Then from Theorem 3.1, Remark 3.3, and Lemma 5.1, we get

$$\begin{aligned}
\mathcal{N}\big(\mathcal{C}_\theta\big(r, 2^{-\beta_{N(r)}}\big)\big) &= n\big(\big[r, 2^{-\beta_{N(r)}}\big[\big) + \mathcal{O}\Big(|\ln\delta|^2 \Big(n([r(1-\delta), r(1+\delta)]) \\
&\qquad + n\big(\big[2^{-\beta_{N(r)}}(1-\delta), 2^{-\beta_{N(r)}}(1+\delta)\big]\big)\Big)\Big) \\
&= n\big(\big[r, 2^{-\beta_{N(r)}}\big[\big) + \mathcal{O}\big(|\ln\delta|^2\big) n([r(1-\delta), r(1+\delta)]) \\
&\qquad + \mathcal{O}\big(\delta|\ln\delta|^2\big) n\big(\big[2^{-N(r)-\frac{1}{4}}, 2^{-N(r)+\frac{1}{4}}\big]\big),
\end{aligned} \tag{5.3}$$

and, for $M(\delta) \leq j \leq N(r) - 1$,

$$\begin{aligned}
\mathcal{N}\big(\mathcal{C}_\theta\big(2^{-\beta_{j+1}}, 2^{-\beta_j}\big)\big) &= n\big(\big[2^{-\beta_{j+1}}, 2^{-\beta_j}\big[\big) + \mathcal{O}\Big(|\ln\delta|^2 \Big(n\big(\big[2^{-\beta_j}(1-\delta), 2^{-\beta_j}(1+\delta)\big]\big) \\
&\qquad + n\big(\big[2^{-\beta_{j+1}}(1-\delta), 2^{-\beta_{j+1}}(1+\delta)\big]\big)\Big)\Big) \\
&= n\big(\big[2^{-\beta_{j+1}}, 2^{-\beta_j}\big[\big) + \mathcal{O}\big(\delta|\ln\delta|^2\big) n\big(\big[2^{-j-\frac{1}{4}}, 2^{-j+\frac{1}{4}}\big]\big) \\
&\qquad + \mathcal{O}\big(\delta|\ln\delta|^2\big) n\big(\big[2^{-j-1-\frac{1}{4}}, 2^{-j-1+\frac{1}{4}}\big]\big).
\end{aligned} \tag{5.4}$$

Moreover, we can write

$$\mathcal{N}\big(\mathcal{C}_\theta\big(2^{-\beta_{M(\delta)}}, 1\big)\big) = n\big(\big[2^{-\beta_{M(\delta)}}, 1\big]\big) + \mathcal{O}_\delta(1). \tag{5.5}$$

Combining (5.2) with (5.3)–(5.5), we deduce

$$\begin{aligned}
\mathcal{N}(\mathcal{C}_\theta(r,1)) &= n([r,1]) + \mathcal{O}\big(|\ln\delta|^2\big) n([r(1-\delta), r(1+\delta)]) \\
&\qquad + \mathcal{O}\big(\delta|\ln\delta|^2\big) \sum_{j=M(\delta)}^{N(r)} n\big(\big[2^{-j-\frac{1}{4}}, 2^{-j+\frac{1}{4}}\big]\big) + \mathcal{O}_\delta(1) \\
&= n([r,1])\big(1 + \mathcal{O}\big(\delta|\ln\delta|^2\big)\big) + \mathcal{O}\big(|\ln\delta|^2\big) n([r(1-\delta), r(1+\delta)]) + \mathcal{O}_\delta(1),
\end{aligned} \tag{5.6}$$

since we have

$$\bigcup_{j=M(\delta)}^{N(r)} \big[2^{-j-\frac{1}{4}}, 2^{-j+\frac{1}{4}}\big] \subset [r,1],$$

the union on the left hand side being disjoint. This concludes the proof of Theorem 3.7. □

In order to prove Corollary 3.9, we need the following

Lemma 5.2. *Let $\Psi :]0,1[\longrightarrow \mathbb{R}$ be a non-increasing function such that $\Psi(r) \geq 1$ and $\Psi(r) = \mathcal{O}(r^{-\gamma})$, $\gamma > 0$, on $]0,1[$. Then, there exists $C > 0$ such that, for any $\delta > 0$ small enough and any $\rho > 0$, there exists $0 < r \leq \rho$ satisfying*

$$\Psi(r(1-\delta)) - \Psi(r(1+\delta)) \leq C\delta\Psi(r).$$

Proof. Assume that the result is not true. Then, for all $C, \delta_1 > 0$, there exists $\rho > 0$ and $0 < \delta < \delta_1$ such that, for all $0 < r \leq \rho$, we have

$$\Psi(r(1-\delta)) - \Psi(r(1+\delta)) \geq C\delta\Psi(r).$$

Changing the variables $r \longmapsto \sigma r$ with $\sigma = \frac{1-\delta}{1+\delta}$, and using the monotonicity and the lower bound of Ψ, we get

$$\Psi(\sigma r) \geq (1 + C\delta)\Psi(r), \quad r \in]0, \rho].$$

Then, for any $K \in \mathbb{N}$,

$$\Psi(\sigma^K \rho) \geq (1 + C\delta)^K \Psi(\rho). \tag{5.7}$$

On the other hand, we have, by assumption,

$$\Psi(\sigma^K \rho) \leq \mathcal{O}(\sigma^{-\gamma K}). \tag{5.8}$$

Since (5.7) and (5.8) hold (uniformly) for all $K \in \mathbb{N}$, we deduce

$$\ln(1 + C\delta) \leq \gamma |\ln \sigma|.$$

Now, letting δ_1 (and, hence, δ) tend to 0, we find that the Taylor expansion in δ yields

$$C \leq 2\gamma,$$

for all $C > 0$. We get a contradiction. □

Proof of Corollary 3.9. We construct the sequence $(r_k)_k$ the following way. Let $\delta > 0$ be small enough such that

$$\mathcal{O}\big(\delta |\ln \delta|^2\big) \leq \frac{1}{k} \qquad \text{and} \qquad C\delta\mathcal{O}\big(|\ln \delta|^2\big) \leq \frac{1}{k}, \tag{5.9}$$

where the $\mathcal{O}$'s are the ones appearing in Theorem 3.7 and C is the constant given in Lemma 5.2. Since $n([r,1]) \to +\infty$ as $r \searrow 0$, one can find $0 < \rho \leq 2^{-k}$ such that

$$\mathcal{O}_\delta(1) \leq \frac{n([\rho, 1])}{k}. \tag{5.10}$$

Now, applying Lemma 5.2 to the function $\Psi(r) := n([r,1])$ with C and δ as before, we deduce that there exists $r_k \leq \rho$ such that

$$n([r_k(1-\delta), r_k(1+\delta)]) \leq C\delta\, n([r_k, 1]). \tag{5.11}$$

By $r_k \in]0, 2^{-k}]$, the positive sequence $(r_k)_{k\in\mathbb{N}}$ tends to 0. Combining estimates (5.9)–(5.11) with Theorem 3.7, we find that

$$\big|\mathcal{N}(\mathcal{C}_\theta(r_k, 1)) - n([r_k, 1])\big| \leq \frac{3}{k} n([r_k, 1]).$$

which implies (3.3). □

Proof of Corollary 3.11. If $n([r,1]) = \Phi(r)(1+o(1))$ with

$$\Phi(r(1\pm\delta)) = \Phi(r)(1+o(1)+\mathcal{O}(\delta)),$$

then $n([r(1-\delta), r(1+\delta)]) = n([r,1])(o(1)+\mathcal{O}(\delta))$. In particular, if in addition $\Phi(r)$ tends to infinity, Theorem 3.7 implies that

$$\mathcal{N}(\mathcal{C}_\theta(r,1)) = \Phi(r)(1+o(1)), \qquad r \searrow 0.$$

Thus, Corollary 3.11 follows from the estimates:

- If $\Phi(r) = r^{-\gamma}$, $\gamma > 0$, then $\Phi(r(1\pm\delta)) = r^{-\gamma}(1\pm\delta)^{-\gamma} = \Phi(r)(1+\mathcal{O}(\delta))$;
- If $\Phi(r) = |\ln r|^\gamma$, $\gamma > 0$, then $\Phi(r(1\pm\delta)) = |\ln r|^\gamma \Big(1 + \frac{\ln(1\pm\delta)}{\ln r}\Big)^\gamma = \Phi(r)(1+o(1))$;
- If $\Phi(r) = \frac{|\ln r|}{\ln|\ln r|}$, then $\Phi(r(1\pm\delta)) = \frac{|\ln r| - \ln(1\pm\delta)}{\ln|\ln r| + \ln\big(1+\frac{\ln(1\pm\delta)}{\ln r}\big)} = \Phi(r)(1+o(1))$. $\square$

6. Application to the counting function of magnetic resonances

In this section, we apply the results of Section 3 to the counting function of resonances of magnetic Schrödinger operators near the Landau levels $2bq$. Let H_0 be the free Hamiltonian defined in (1.2) and (1.3). The selfadjoint operator H_0 is first defined on $C_0^\infty(\mathbb{R}^3)$, and then is closed in $L^2(\mathbb{R}^3)$. On the domain of H_0, we introduce $H := H_0 + V$ where $V : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is an appropriate electric potential. More precisely, we assume that V is Lebesgue measurable, and satisfies (1.8) for some $m_\perp > 0$ and $N > 0$. Under this assumption, the operator H is selfadjoint with essential spectrum $[0, +\infty[$, and its resonances near the real axis are defined as the poles of the meromorphic extension of the resolvent $z \longmapsto (H-z)^{-1}$ considered as an element of $\mathcal{L}(e^{-N\langle x_3\rangle} L^2(\mathbb{R}^3_{\mathbf{x}}), e^{N\langle x_3\rangle} L^2(\mathbb{R}^3_{\mathbf{x}}))$ (for more details see [7]). In Proposition 6.2 below, we describe a useful local characterization of the resonances of $H(b,V)$ near a given Landau level $2bq$, $q \in \mathbb{N}$. First, near $2bq$, we parametrize z by $2bq + k^2$, and we have

Proposition 6.1 ([7, Lemma 1])**.** *For V satisfying* (1.8) *and $q \in \mathbb{N}$, the operator valued function*

$$k \longmapsto T_{V,q}(k) := J|V|^{\frac{1}{2}}\big(H_0 - 2bq - k^2\big)^{-1}|V|^{\frac{1}{2}}, \qquad J := \operatorname{sign} V,$$

defined in $]0, \sqrt{2b}[e^{i]0,\pi/2[}$, has an analytic extension to the set $\mathcal{D}\backslash\{0\}$ where $\mathcal{D} := \{k \in \mathbb{C};\ 0 \leq |k| < \min(\sqrt{2b}, N)\}$.

Then, using the resolvent equation

$$\big(I - J|V|^{\frac{1}{2}}(H-z)^{-1}|V|^{\frac{1}{2}}\big)\big(I + J|V|^{\frac{1}{2}}(H_0-z)^{-1}|V|^{\frac{1}{2}}\big) = I, \tag{6.1}$$

we obtain the desired characterization of the resonances of H:

Proposition 6.2. *Under assumption* (1.8), *$z_0 = 2bq + k_0^2$ is a resonance of H near $2bq$, $q \in \mathbb{N}$, if and only if k_0 is a characteristic value of $I + T_{V,q}(\cdot)$, and the multiplicity of this resonance coincides with the multiplicity of the characteristic value defined in Definition 2.4.*

Proof. If $m_\perp > 2$, Proposition 6.2 follows immediately from [7, Proposition 3] and (2.6). If $m_\perp \in]0, 2]$, the same proof works using $\det_p$ with $p > 2/m_\perp$. $\square$

In order to formulate our further results, we need the following notations. Let p_q be the orthogonal projection onto $\ker(H_{\text{Landau}} - 2bq)$, the Landau Hamiltonian H_{Landau} being defined in (1.4). The operator p_q admits an explicit kernel

$$\mathcal{P}_{q,b}(X_\perp, X'_\perp) = \frac{b}{2\pi} L_q\left(\frac{b|X_\perp - X'_\perp|^2}{2}\right) \exp\Big(-\frac{b}{4}\big(|X_\perp - X'_\perp|^2 + 2i(x_1x'_2 - x'_1x_2)\big)\Big),$$

with $X_\perp, X'_\perp \in \mathbb{R}^2$; here $L_q(t) := \frac{1}{q!} e^t \frac{d^q(t^q e^{-t})}{dt^q}$ are the Laguerre polynomials. Further, we recall that I_3 is the identity operator in $L^2(\mathbb{R}_{x_3})$. Finally, we denote by $r(z)$ an operator with integral kernel $\frac{1}{2} e^{z|x_3 - x'_3|}$, $x_3, x'_3 \in \mathbb{R}$, depending on the parameter $z \in \mathbb{C}$.

The following proposition shows that we are in the framework of Section 3.

Proposition 6.3 ([7, Proposition 4]). *Assume that V satisfies* (1.8) *and fix $q \in \mathbb{N}$. Then for $k \in \mathcal{D} \setminus \{0\}$, we have*

$$I + T_{V,q}(k) = I - \frac{A_q(ik)}{ik},$$

where $z \longmapsto A_q(z) \in \mathcal{S}_\infty(L^2(\mathbb{R}^3))$ is the holomorphic function given by

$$A_q(z) = J|V|^{\frac{1}{2}} p_q \otimes r(z) |V|^{\frac{1}{2}} - zJ \sum_{j \neq q} |V|^{\frac{1}{2}} (p_j \otimes I_3) \big(D_3^2 + 2b(j-q) + z^2 \big)^{-1} |V|^{\frac{1}{2}}. \tag{6.2}$$

Consequently, the resonances of H near a fixed Landau level $2bq$ coincide with the complex number $2bq + k^2$ where k is a characteristic value of $\big(I - \frac{A_q(ik)}{ik} \big)$ and A_q is given by (6.2). In particular, $A_q(0)$ is the operator $J|V|^{\frac{1}{2}} (p_q \otimes r(0)) |V|^{\frac{1}{2}}$ which is selfadjoint as soon as J is $\pm I$, i.e. for V of definite sign.

Now we assume that V has a definite sign, i.e. $\pm V \geq 0$. In order to apply the results of Section 3, we want to know when

$$I - A'_q(0)\Pi_0 \text{ is invertible,} \tag{6.3}$$

where, as earlier, Π_0 is the orthogonal projection on the kernel of $A_q(0)$. Writing $A_q(0) = \pm L_q^* L_q$ with $L_q : L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^2)$ defined by

$$(L_q f)(X_\perp) := \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3_{\mathbf{x}}} \mathcal{P}_{q,b}(X_\perp, X'_\perp) |V|^{\frac{1}{2}} (X'_\perp, x'_3) f(X'_\perp, x'_3) \, dX'_\perp \, dx'_3, \quad X_\perp \in \mathbb{R}^2, \tag{6.4}$$

we find that $\ker A_q(0) = \ker L_q$.

Remark 6.4. *In general,* $\ker A_q(0) = \ker L_q$ *is not trivial. Nevertheless, the assumption* (6.3) *holds for generic V. More precisely, if the potential V is fixed, there exists a finite or infinite discrete set $\mathcal{E} = \{e_n\}$ such that the operator $H_e := H_0 + eV$ satisfies* (6.3) *for all $e \in \mathbb{R} \setminus \mathcal{E}$. The numbers $1/e_n$ are in fact the real non vanishing eigenvalues of the compact operator $A'_q(0)\Pi_0$. To check this, it is enough to remark that $\Pi_{0|_{eV}} = \Pi_{0|_V}$ and $A'_q(0)_{|_{eV}} = eA'_q(0)_{|_V}$ for $e \neq 0$. Note also that, for $|e|$ small enough, H_e satisfies always* (6.3).

Under these assumptions, we can apply Theorem 3.7 and its corollaries. Thus, the distribution of the magnetic resonances near the Landau level is related to the counting function

$$n_\pm(s; A_q(0)) = n_+(s; L_q^* L_q) = n_+(s; L_q L_q^*) = n_+(s; p_q W p_q),$$

where, for a compact selfadjoint operator T, we set $n_\pm(s;T) = \operatorname{rank} \mathbb{1}_{\pm[s,+\infty[}(T)$, and W is the multiplication operator by the function

$$W(X_\perp) := \frac{1}{2} \int_{\mathbb{R}} |V(X_\perp, x_3)| \, dx_3, \qquad X_\perp \in \mathbb{R}^2. \tag{6.5}$$

Let us introduce three types of assumptions for W:

(A1) $W \in C^1(\mathbb{R}^2)$ satisfies the estimate

$$W(X_\perp) = w_0(X_\perp/|X_\perp|)|X_\perp|^{-m_\perp}(1 + o(1)), \qquad |X_\perp| \to +\infty,$$

where w_0 is a continuous function on $\mathbb{S}^1$ which is non-negative and does not vanish identically, as well as

$$|\nabla W(X_\perp)| \leq C\langle X_\perp\rangle^{-m_\perp-1}, \qquad X_\perp \in \mathbb{R}^2,$$

for some constant $C > 0$. Then, by [29], we have

$$n_+(r, p_q W p_q) = C_\perp r^{-2/m_\perp}(1 + o(1)), \qquad r \searrow 0, \tag{6.6}$$

where

$$C_\perp := \frac{b}{4\pi}\int_{\mathbb{S}^1} w_0(t)^{2/m_\perp}\, dt.$$

(A2) There exists $\beta > 0$, $\mu > 0$ such that

$$\ln W(X_\perp) = -\mu|X_\perp|^{2\beta}(1 + o(1)), \qquad |X_\perp| \to +\infty.$$

Then, by [33], we have

$$n_+(r, p_q W p_q) = \varphi_\beta(r)(1 + o(1)), \qquad r \searrow 0, \tag{6.7}$$

where, for $0 < r \ll 1$,

$$\varphi_\beta(r) := \begin{cases} \frac{b}{2}\mu^{-\frac{1}{\beta}}|\ln r|^{\frac{1}{\beta}} & \text{if } 0 < \beta < 1, \\ \frac{1}{\ln(1 + 2\mu/b)}|\ln r| & \text{if } \beta = 1, \\ \frac{\beta}{\beta - 1}(\ln|\ln r|)^{-1}|\ln r| & \text{if } \beta > 1. \end{cases}$$

(A3) The support of W is compact and there exists a constant $C > 0$ such that $W \geq C$ on an non-empty open subset of $\mathbb{R}^2$. Then, by [33], we have

$$n_+(r, p_q W p_q) = \varphi_\infty(r)(1 + o(1)), \qquad r \searrow 0, \tag{6.8}$$

where, for $0 < r \ll 1$,

$$\varphi_\infty(r) := (\ln|\ln r|)^{-1}|\ln r|.$$

In particular, $n_+(r, p_q W p_q) \longrightarrow +\infty$ as $r \searrow 0$, provided that V does not vanish identically.

Theorem 6.5. *Let $V : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a Lebesgue measurable function of definite sign $\pm$ satisfying* (1.8) *and* (6.3). *Let $0 < r_0 < \min(\sqrt{2b}, N)$ be fixed. Then,*

i) The resonances $z_q(k) = 2bq + k^2$ of $H = H_0 + V$ with $|k|$ sufficiently small satisfy

$$\pm \operatorname{Im} k \leq 0, \qquad \operatorname{Re} k = o(|k|).$$

ii) There exists a sequence $(r_\ell)_\ell \in \mathbb{R}$ which tends to 0 such that

$$\#\big\{z = 2bq + k^2 \in \operatorname{Res}(H);\ r_\ell < |k| \leq r_0\big\} = n_+(r_\ell, p_q W p_q)(1 + o(1)), \qquad \ell \to +\infty.$$

iii) Eventually, if W satisfies (A1), (A2) *or* (A3), *then*

$$\#\big\{z = 2bq + k^2 \in \operatorname{Res}(H);\ r < |k| \leq r_0\big\} = n_+(r, p_q W p_q)(1 + o(1)), \qquad r \searrow 0,$$

the asymptotics of $n_+(r, p_q W p_q)$ as $r \searrow 0$ being described in (6.6)–(6.8).

Remark 6.6. *i) Under the assumption* (A1), *we can also apply Theorem 3.1 to obtain asymptotics in small domains.*

ii) In [7], *we have proved that H has an infinite number of resonances in a vicinity of* 0 *for small potentials V of definite sign such that W, defined in* (6.5), *satisfies, for some $C > 0$,*

$$\ln W(X_\perp) \leq -C\langle X_\perp\rangle^2.$$

iii) These results can be generalized to the case of constant magnetic fields of non full rank $2r$ in an arbitrary dimension d. More precisely, the situation $d - 2r = 1$ is close to the one treated in the present paper. Whereas, if $d - 2r \geq 3$ is odd, it is expected that there is no accumulation of resonances at the Landau levels since the corresponding $A(z)$ is analytic near these thresholds. The case $d - 2r$ even is different since the weighted resolvent has a logarithmic singularity at the Landau levels.

Proof of Theorem 6.5. According to Definition 6.2 and Proposition 6.3, in order to study the resonances $z_q(k) = 2bq+k^2$ of H, it is enough to analyze the characteristic values of $\big(I-\frac{A_q(ik)}{ik}\big)$ for A_q given by (6.2). Since $\pm A_q(0)$ is non negative, *i*) is a consequence of Corollary 3.4 with $z = ik$.

From *i*) we deduce that the resonances $z_q(k) = 2bq + k^2$ are concentrated in the sector $\mp i\mathcal{C}_\theta \cap \mathcal{D}$ for every $\theta > 0$ with $\mathcal{C}_\theta$ defined by (3.1). In particular, as r tends to 0, we have:

$$\begin{aligned}\#\big\{z = 2bq + k^2 \in \operatorname{Res}(H);\ r_\ell < |k| \leq r_0\big\}\\ = \#\big\{z = 2bq + k^2 \in \operatorname{Res}(H);\ \pm ik \in \mathcal{C}_\theta(r, r_0)\big\} + \mathcal{O}(1).\end{aligned}$$

Since the non-zero eigenvalues of $\pm A_q(0) = L_q^* L_q$, L_q being defined in (6.4), coincide with these of $L_q L_q^* = p_q W p_q$, we have $n([r, r_0]) = n_+(r, p_q W p_q) + \mathcal{O}(1)$. Then, parts *ii*) and *iii*) follow from Corollary 3.9 and Corollary 3.11. □

Corollary 6.7. *For generic potentials $V \geq 0$ satisfying* (1.8), *the (embedded) eigenvalues of H form a discrete set.*

This is a consequence of Remark 6.4 and Theorem 6.5 *i*) at each Landau level. In [7, Proposition 7], it is proved that, for small potentials $V \geq 0$ satisfying (1.7) with $m_\perp > 0$ and $m_3 > 2$, there are no eigenvalues outside of the Landau levels $2b\mathbb{N}$. Recall that the setting is very different for non-positive perturbations. Indeed, for a large class of non-positive potentials, there is an accumulation of embedded eigenvalues at each Landau level (see (1.6) and the references [30], [31]). Note that, using the Mourre theory, it should be possible to show that the eigenvalues may accumulate only at the Landau levels (see for instance [14, Theorem 3.5.3] in a slightly different general context).

7. Necessity of the assumptions of the main results

In this section we show that all the assumptions of the Theorem 3.1 and Theorem 3.7 are necessary for the claimed properties in the sense that these results do not hold if one removes one of their hypotheses (it is perhaps possible to consider other types of assumptions). An artificial reason would be easily given by examples for which the characteristic values are not well defined (for instance when $I - \frac{A(z)}{z}$ is never invertible). However, the proposition below shows that there are some more fundamental obstructions to the weakening of the assumptions.

Proposition 7.1. *Even if the assumptions of Proposition 2.3 are satisfied, the conclusions of the theorems of Section 3 may be false if one of the following hypotheses is removed:*
i) $A(z)$ *is compact-valued;*
ii) $A(0)$ *is selfadjoint;*
iii) $I - A'(0)\Pi_0$ *is invertible.*

Proof. *i)* If $A(z) = A(0)+A'(0)z$ where $A(0)$ is a selfadjoint, compact operator with $\ker A(0) = \{0\}$ and $A'(0) = 2I$, then the assumptions of Proposition 2.3, Theorem 3.1 and Theorem 3.7, except the compactness of $A(z)$, hold true. In this case, the characteristic values are well defined and given by $(-\lambda_k)_{k\geq 1}$ where $(\lambda_k)_{k\geq 1}$ are the eigenvalues of $A(0)$. Thus, the counting function of the eigenvalues of $A(0)$ and the counting function of the characteristic values of $I - \frac{A(z)}{z}$ are very different.

ii) On $\mathcal{H} = \ell^2(\mathbb{N})$, let us consider the infinite block diagonal matrices

$$A(0) = \mathrm{diag}(B_0, \dots, B_k, \dots) \qquad \text{and} \qquad A'(0) = \mathrm{diag}(B_0', \dots, B_k', \dots),$$

where B_k, B_k', $k \in \mathbb{N}$, are the following 2×2 matrices

$$B_k = \begin{pmatrix} 0 & 0 \\ \alpha_k & 0 \end{pmatrix} \qquad \text{and} \qquad B_k' = \begin{pmatrix} 0 & \alpha_k \\ 0 & 0 \end{pmatrix},$$

with $(\alpha_k)_{k\in\mathbb{N}}$ a sequence of real numbers which tends to 0. Then, the characteristic values of $I - \frac{A(0)}{z} - A'(0)$ are the complex numbers z for which one of the matrices

$$I - \frac{B_k}{z} - B_k' = \begin{pmatrix} 1 & -\alpha_k \\ -\frac{\alpha_k}{z} & 1 \end{pmatrix},$$

is not invertible. Thus, these characteristic values are the real numbers α_k^2, $k \in \mathbb{N}$, while the spectrum of $A(0)$ is reduced to $\{0\}$ since $A(0)$ is nilpotent. Note that $I - A'(0)\Pi_0 = \mathrm{diag}(I - B_0', \dots, I - B_k', \dots)$ is invertible. So, we have an example where all the assumptions of Proposition 2.3, Theorem 3.1 and Theorem 3.7, except the selfadjointness of $A(0)$, are met, but the conclusions of the main theorems of Section 3 do not hold true.

iii) On $\mathcal{H} = \mathbb{C} \oplus \ell^2(\mathbb{N})$, let us consider the affine function

$$A(z) = A(0) + A'(0)z \tag{7.1}$$

with

$$A(0) = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{diag}(-\lambda_k) \end{pmatrix} \qquad \text{and} \qquad A'(0) = \begin{pmatrix} 1 & \alpha \\ {}^t\alpha & 0 \end{pmatrix}, \tag{7.2}$$

where $\alpha = (\alpha_0, \dots, \alpha_k, \dots) \in \ell^2(\mathbb{N})$ and $(\lambda_k)_{k\in\mathbb{N}}$ goes to 0. Let

$$f_n(z) = \sum_{k=0}^{n} \alpha_k^2 \Big(\frac{\lambda_k}{z} + 1 \Big)^{-1} \qquad \text{and} \qquad f_\infty(z) = \sum_{k=0}^{+\infty} \alpha_k^2 \Big(\frac{\lambda_k}{z} + 1 \Big)^{-1}.$$

We construct inductively two sequences $(\lambda_k)_{k\in\mathbb{N}}$ and $(\alpha_k)_{k\in\mathbb{N}}$ for which we have

$$(H)_n : \begin{cases} 0 < \lambda_n < \cdots < \lambda_0, \\ \forall 0 \leq k \leq n \quad 0 < |\alpha_k| \leq 2^{-k} \text{ and } \alpha_k^2 \in \mathbb{R}, \\ \forall 0 \leq k \leq n \quad (-1)^k f_n(\lambda_k) \geq \dfrac{|\alpha_k|^2}{4}\Big(1 + \dfrac{1}{n+1}\Big), \end{cases}$$

for all $n \in \mathbb{N}$. One can verify that $(H)_0$ holds with $\lambda_0 = \alpha_0 = 1$.

Lemma 7.2. *If $(H)_n$ holds, there exist λ_{n+1} and α_{n+1} such that $(H)_{n+1}$ holds.*

Proof of Lemma 7.2. We choose

$$\alpha_{n+1} = i^{n+1} \min \Big(2^{-n-1}, \min_{0 \leq k \leq n} \frac{|\alpha_k|}{2} \Big(\frac{1}{n+1} - \frac{1}{n+2} \Big)^{\frac{1}{2}} \Big).$$

In particular, $0 < |\alpha_{n+1}| \leq 2^{-n-1}$, $\alpha_{n+1}^2 \in \mathbb{R}$ and

$$\begin{aligned} (-1)^k f_{n+1}(\lambda_k) &= (-1)^k f_n(\lambda_k) + (-1)^k \alpha_{n+1}^2 \Big(\frac{\lambda_{n+1}}{\lambda_k} + 1 \Big)^{-1} \\ &\geq \frac{|\alpha_k|^2}{4} \Big(1 + \frac{1}{n+1} \Big) - \frac{|\alpha_k|^2}{4} \Big(\frac{1}{n+1} - \frac{1}{n+2} \Big) \\ &= \frac{|\alpha_k|^2}{4} \Big(1 + \frac{1}{(n+1)+1} \Big), \end{aligned}$$

for $0 \leq k \leq n$. It remains to evaluate

$$\begin{aligned} (-1)^{n+1} f_{n+1}(\lambda_{n+1}) &= (-1)^{n+1} f_n(\lambda_{n+1}) + |\alpha_{n+1}|^2 \Big(\frac{\lambda_{n+1}}{\lambda_{n+1}} + 1 \Big)^{-1} \\ &= \frac{|\alpha_{n+1}|^2}{2} + (-1)^{n+1} f_n(\lambda_{n+1}). \end{aligned}$$

Note that $f_n(x) \longrightarrow 0$ as $x \searrow 0$. Then, one can choose $0 < \lambda_{n+1} < \lambda_n$ small enough such that $|f_n(\lambda_{n+1})| \leq \frac{|\alpha_{n+1}|^2}{4}$. So, the previous equation implies

$$(-1)^{n+1} f_{n+1}(\lambda_{n+1}) \geq \frac{|\alpha_{n+1}|^2}{4},$$

and $(H)_{n+1}$ holds. □

Using Lemma 7.2, we can construct $(\lambda_k)_{k \in \mathbb{N}}$ and $(\alpha_k)_{k \in \mathbb{N}}$ such that $(H)_n$ holds for all $n \in \mathbb{N}$. With such a choice, the operators $A(0)$ and $A'(0)$ in (7.1)–(7.2) are compact, and $A(0)$ is selfadjoint. However, the operator

$$I - A'(0) \Pi_0 = \begin{pmatrix} 0 & 0 \\ -{}^t\alpha & 1 \end{pmatrix},$$

is not invertible. On the other hand, $f_\infty(z)$ is a well defined holomorphic function for $z \in \mathbb{C} \setminus \{0\} \cup \{-\lambda_k;\ k \in \mathbb{N}\}$. Moreover, $f_n \longrightarrow f_\infty$ uniformly on the compact subset of $\mathbb{C} \setminus \{0\} \cup \{-\lambda_k;\ k \in \mathbb{N}\}$. In particular, this implies that

$$(-1)^k f_\infty(\lambda_k) \geq \frac{|\alpha_k|^2}{4} > 0, \tag{7.3}$$

for all $k \in \mathbb{N}$.

Lemma 7.3. *For $z \in \mathbb{C} \setminus \{0\} \cup \{-\lambda_k;\ k \in \mathbb{N}\}$, we have*

$$I - \frac{A(z)}{z} \text{ is not invertible} \quad \Longleftrightarrow \quad f_\infty(z) = 0.$$

Proof of Lemma 7.3. This result is a direct consequence of the invertibility of $D = 1 + \frac{\operatorname{diag}(\lambda_k)}{z}$ and the identity

$$\begin{pmatrix} 1 & \alpha D^{-1} \\ 0 & D^{-1} \end{pmatrix} \Big(I - \frac{A(z)}{z} \Big) = \begin{pmatrix} 1 & \alpha D^{-1} \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} 0 & -\alpha \\ -{}^t\alpha & D \end{pmatrix}$$
$$= \begin{pmatrix} -\alpha D^{-1t}\alpha & 0 \\ -D^{-1t}\alpha & 1 \end{pmatrix} = \begin{pmatrix} -f_\infty(z) & 0 \\ -D^{-1t}\alpha & 1 \end{pmatrix}.$$

□

Combining the previous lemma with (7.3), we find that $I - \frac{A(z)}{z}$ is invertible on the λ_k, and then the assumptions of Proposition 2.3 hold. So, the characteristics values in $\mathbb{C} \setminus \{0\} \cup \{-\lambda_k;\ k \in \mathbb{N}\}$ are well defined and coincide with the zeroes of f_∞. On the other hand, it follows from $(H)_n$ and (7.3) that $f_\infty(x)$ is a continuous real-valued function on $]0, +\infty[$ which changes its sign between λ_{k+1} and λ_k. Then the intermediate value theorem implies that, for all $k \in \mathbb{N}$, there exists a characteristic value x_k with $0 < \lambda_{k+1} < x_k < \lambda_k$. At the same time, the eigenvalues of $A(0)$ are the $-\lambda_k$ which are all negative.

Summing up, we have constructed an example where all the assumptions of Proposition 2.3, Theorem 3.1 and Theorem 3.7, except the invertibility of $I - A'(0)\Pi_0$, are met, but the conclusions of the main theorems of Section 3 do not hold true. □

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